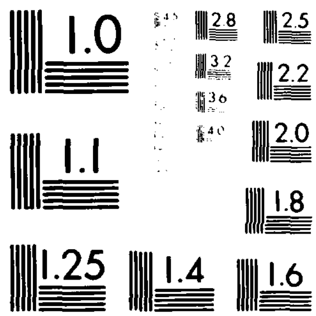
AD-A083 265 DELAWARE UNIV NEWARK APPLIED MATHEMATICS INST F/G 20/11
A BOUNDARY-LAYER THEORY FOR THE ANISOTROPIC PLATE.(U)
MAR 80 R P GILBERT, M SCHNEIDER AFOSR-76-2879
UNCLASSIFIED TR-71A AFOSR-TR-80-0297 NL

END
DATE FILMED
5-80
DTIC



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS 1963 A

AFOSR-TR- 80-0297

7

LEVEL II

ADA083265

APPLIED MATHEMATICS INSTITUTE

University of Delaware
Newark, Delaware

DTIC
ELECTE
APR 21 1980
S D
E

Approved for public release;
distribution unlimited.

80 4 21 011

DDC FILE COPY

A Boundary-Layer Theory for the Anisotropic Plate*

by

R. P. Gilbert

and

M. Schneider
Universität Karlsruhe

Applied Mathematics Institute
Technical Report No. 71A

A. D. [illegible]
Technical Information Officer

*This work was supported in part by the Air Force Office of Scientific Research through Grant [redacted]AFOSR76-2879.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER (18) AFOSR-TR-80-0297 (19)

2. GOVT ACCESSION NO. AD-A083 265

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) (6) A BOUNDARY-LAYER THEORY FOR THE ANISOTROPIC PLATE.

5. TYPE OF REPORT & PERIOD COVERED (9) Interim Rept.,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) (10) R. P. Gilbert [redacted] M. Schneider

8. CONTRACT OR GRANT NUMBER(s) (15) AFOSR-76-2879

9. PERFORMING ORGANIZATION NAME AND ADDRESS
University of Delaware
Applied Mathematics Institute
Newark, DE 19711

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) (17)
61102F 2304/A4

11. CONTROLLING OFFICE NAME AND ADDRESS
Air Force Office Of Scientific Research/NM
Bolling AFB, Washington, DC 20332

12. REPORT DATE (11) Mar[redacted]80

13. NUMBER OF PAGES 27

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 29

15. SECURITY CLASS. (of this report) UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(14) TR-71A

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Accession For	
NTIS GRA&I	✓
DDC TAB	
Unannounced	
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

To our knowledge the results of the pioneering paper on the boundary layer theory for plates by K. O. Friedrichs and R. F. Dressler have not been extended to anisotropic plates. Yet the basic conclusion of the above paper, namely that the equations of three dimensional isotropic elasticity can be split into two independent systems seems to have been accepted by some authors to include also the case of anisotropic elasticity. While this happens to be true it is by no means obvious as the following presentation illustrates; hence, we believe our providing a detailed analysis of the orthotropic plate will be of interest to -

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 411278

20. Abstract cont.

researchers in the field of composite materials and laminates.

Introduction

To our knowledge the results of the pioneering paper on the boundary layer theory for plates by K. O. Friedrichs and R. F. Dressler [1] have not been extended to anisotropic plates.† Yet the basic conclusion of the above paper, namely that the equations of three dimensional isotropic elasticity can be split into two independent systems seems to have been accepted by some authors to include also the case of anisotropic elasticity [7] [ ]. While this happens to be true it is by no means obvious as the following presentation illustrates; hence, we believe our providing a detailed analysis of the orthotropic plate will be of interest to researchers in the field of composite materials and laminates.

We point out that at the *zeroth order of approximation of* the interior solution we obtain the expected thin, orthotropic plate equation for the stress function, namely

$$a_{22}\, a_{66}\, W^{(o)}_{,xxxx} + 2\,(2a_{11}\, a_{22} - 2a_{12}^2 - a_{12}\, a_{66})\, W^{(o)}_{,xxyy}$$

$$+\, a_{11}\, a_{66}\, W^{(o)}_{,yyyy} = -\frac{3}{2}\, a_{66}\,(a_{11}\, a_{22} - a_{12}^2)\, p^{(3)}\,(x,y)$$

For the boundary layer approximation in the limit domain we find at the h^1-step that the stress function satisfies an equation having a three-dimensional character in that the coefficients depend on the Z-direction

† From personal correspondence with Professor Friedrichs we have learned that he also is unaware of any generalizations of this type.

$$\frac{1}{E_3}(1-\nu_{23}\nu_{32})\theta,_{\S\S\S\S} + \left(-\frac{2\nu_{13}}{E_1} + \frac{1}{G_{13}} - \frac{2}{E_1}\nu_{12}\nu_{23}\right)\theta,_{\S\S\S\S}$$

$$+ \frac{1}{E_1}(1-\nu_{12}\nu_{21})\theta,_{\S\S\S\S} = 0.$$

Using typical values[+] for a highly anisotropic material, namely $E_1 = 20 \times 10^6$ psi, $E_2 = 1.6 \times 10^6$ psi, $E_3 = 1.3 \times 10^6$ psi, $G_{12} = 0.6 \times 10^6$ psi, $G_{13} = 0.6 \times 10^6$ psi, $G_{23} = 0.5 \times 10^6$ psi. $\nu_{12} = \nu_{13} = \nu_{23} = 0.3$ we compute the above coefficients to be

$$a: = \frac{1}{E_3}(1-\nu_{23}\nu_{32}) = 0.7 \times 10^{-6}$$

$$2b: = \left(-\frac{2\nu 13}{E_1} + \frac{1}{G_{13}} - \frac{2}{E_1}\nu_{12}\nu_{23}\right) = 1.6 \times 10^{-6}$$

$$c: = \frac{1}{E_1}(1-\nu_{12}\nu_{21}) = 0.046 \times 10^{-6}$$

These coefficients show that the equation is elliptic as the roots of the equation $a\mu^4+2b\mu^2+c = 0$ are seen to be $\mu_{1,3} \cong + i\ (0.73)$ and $\nu_{2,4} \cong \pm i\ (1.66)$. In this case, a complex solution may be sought in the form [3] (pp. 29-30)

$$F(x,y) = F_1(x+\mu_1 y) + F_2(x+\mu_2 y) + F_3\ (x+\bar{\mu}_1 y) + F_4(x+\bar{\mu}_2 y)$$

which may be rewriteen for real solutions as

$$F(x,y) = 2\ \mathrm{Re}\left\{F_1(Z_1) + F_2(Z_2)\right\}$$

† Provided to us by Professor R. B. Pipes, Director of the Composite Materials Center, University of Delaware.

where $Z_1 := x+\mu_1 y$, $Z_2 := x+\mu_2 y$. Consequently, it is not technically difficult to find solutions for the boundary layer solution using function theoretic methods.

Various studies on interlaminate stresses have been made by various authors [2], [4-7]. Most of these essentially avoid the anisometric nature of the different laminates. It would be interesting to attempt a complete analysis of the laminate problem bases on the results of the present work.

We consider in this work the case of an orthotropic elastic body, namely one which has three planes of elastic symmetry. Furthermore, we shall assume that these planes are mutually perpendicular and that they are orthogonal to the axes of a Cartesian coordinate system. The generalized Hooke's law for such a body may be written as [3] pg. 8.

$$\varepsilon_x := u_{,x} = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z \tag{1a}$$

$$\varepsilon_y := v_{,y} = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z \tag{1b}$$

$$\varepsilon_z := w_{,z} = a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z \tag{1c}$$

$$\gamma_{yz} := v_{,z} + w_{,y} = a_{44}\tau_{yz} \tag{1d}$$

$$\gamma_{xz} := w_{,x} + u_{,z} = a_{55}\tau_{xz} \tag{1e}$$

$$\gamma_{xy} := u_{,y} + v_{,x} = a_{66}\tau_{xy} \tag{1f}$$

The elastic constants a_{ij} may be expressed in terms of the engineering constants, E_{ij} (i=1,2,3) (Young's moduli), ν_{ij} (i,j=1,2,3) (Poisson's ratio), and the G_{ij} (i,j=1,2,3) (shear moduli) as follows:

$$\begin{aligned} &a_{11} = \frac{1}{E_1}, \quad a_{12} = -\frac{\nu_{21}}{E_2}, \quad a_{13} = -\frac{\nu_{31}}{E_3}, \\ &a_{22} = \frac{1}{E_2}, \quad a_{23} = -\frac{\nu_{32}}{E_3}, \quad a_{33} = \frac{1}{E_3}, \\ &a_{44} = \frac{1}{G_{23}}, \quad a_{55} = \frac{1}{G_{13}}, \quad a_{66} = \frac{1}{G_{12}}; \end{aligned} \tag{2}$$

Furthermore,

$$E_1\nu_{21} = E_2\nu_{12}, \; E_2\nu_{32} = E_3\nu_{23}, \; E_3\nu_{13} = E_1\nu_{31}.$$

When there is a plane of isotropy then the elastic constants simplify, as

$$a_{11} = \frac{1}{E}, \; a_{12} = -\frac{\nu}{E}, \; a_{13} = -\frac{\nu'}{E'},$$

$$a_{22} = \frac{1}{E}, \; a_{23} = -\frac{\nu'}{E'}, \; a_{33} = \frac{1}{E'}, \tag{3}$$

$$a_{44} = \frac{1}{G}, \; a_{55} = \frac{1}{G'}, \; a_{66} = \frac{1}{G},$$

where

$$G = \frac{E}{2(1+\nu)}, \quad \text{and} \quad G' = \frac{E'}{2(1+\nu')} .$$

The problem studied by Friedrichs and Dressler [ ] involved an isotropic body, namely the case of complete symmetry where $E=E'$ and $\nu=\nu'$. In what follows we shall show that their analysis for the isotropic plate may be generalized to the case of the orthotropic plate.

We consider an elastic plate of uniform thickness 2h, lying parallel to one plane of elastic symmetry. The plate shall have an edge which is defined by a continous curve B possessing a continuous tangent. The weight of the plate and other body forces are to be neglected. As in [1] we consider the plate to be deformed by an arbitrary system of normal stresses distributed over the lateral forces, and an arbitrary distribution of normal and shear stresses which vary along the generators and perimeter. Adopting the notation of [1] we consider the following loads to be applied to the plate

$$\bar{\sigma}_z(x,y,h) = -a(x,y) \quad \text{top face}$$

$$\bar{\sigma}_z(x,y,-h) = -b(x,y) \quad \text{bottom face} \tag{4}$$

$$\bar{\tau}_{xz}(x,y,\pm h) = \bar{\tau}_{yz}(x,y,\pm h) = 0.$$

The stresses around the edge are given by $\bar{\sigma}_n(s,z)$, $\bar{\tau}_{ns}(s,z)$, $\bar{\tau}_{nz}(s,z)$, where $\bar{\tau}_{nz}(s,\pm h)=0$ for consistency.

The full three-dimensional system of equations which the plate must satisfy are given by $(\tau_{ij}=\tau_{ji})$,

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} = 0 \tag{5a}$$

$$\tau_{xy,x} + \sigma_{y,y} + \tau_{yz,z} = 0 \tag{5b}$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0 \tag{5c}$$

and the equations (1a-1f). The equations (1a-1f) involving the displacements may be replaced by six compatability equations

$$-a_{44}\sigma_{x,xx}+2a_{13}\sigma_{x,yy}+2a_{12}\sigma_{x,zz}+(2a_{23}+a_{44})\sigma_{y,yy}+2a_{22}\sigma_{y,zz}$$
$$+2a_{33}\sigma_{z,yy}+(2a_{23}+a_{44})\sigma_{z,zz} = 0, \tag{6a}$$

$$(2a_{13}+a_{55})\sigma_{x,xx}+2a_{11}\sigma_{x,zz}+2a_{23}\sigma_{y,xx}-a_{55}\sigma_{y,yy}+2a_{12}\sigma_{y,zz}$$
$$+2a_{33}\sigma_{z,xx}+(2a_{13}+a_{55})\sigma_{z,zz} = 0, \tag{6b}$$

$$(2a_{12}+a_{66})\sigma_{x,xx}+2a_{11}\sigma_{x,yy}+2a_{22}\sigma_{y,xx}+(2a_{12}+a_{66})\sigma_{y,yy}+2a_{23}\sigma_{z,xx}$$
$$+2a_{13}\sigma_{z,yy}-a_{66}\sigma_{z,zz} = 0, \tag{6c}$$

$$[2a_{11}\sigma_x+(2a_{12}+a_{66})\sigma_y+(2a_{13}+a_{55})\sigma_z]_{,yz}+a_{44}\tau_{yz,xx}+a_{55}\tau_{yz,yy}$$
$$+a_{66}\tau_{yz,zz} = 0, \tag{6d}$$

$$[(2a_{12}+a_{66})\sigma_x+2a_{22}\sigma_y+(2a_{23}+a_{44})\sigma_z]_{,xz}+a_{44}\tau_{xz,xx}+a_{55}\tau_{xz,yy}$$
$$+a_{66}\tau_{xz,zz} = 0, \tag{6e}$$

$$[(2a_{13}+a_{55})\sigma_x+(2a_{23}+a_{44})\sigma_y+2a_{33}\sigma_z]_{,xy}+a_{44}\tau_{xy,xx}+a_{55}\tau_{xy,yy}$$
$$+a_{66}\tau_{xy,zz} = 0. \tag{6f}$$

The thickness of the plate is measured in the z-direction. Following [1] we split the stresses applied around the edge into even and odd parts with respect to the variable z, namely

$\bar{\sigma}_n = \bar{\sigma}_n^{(e)} + \bar{\sigma}_n^{(0)}$, etc.; similarly the face conditions are rewritten in terms of "even" and "odd" functions as

$$\bar{\sigma}_z^{(0)}(x,y,h) = -\frac{p(x,y)}{2} = -\frac{a}{2} + \frac{b}{2}\ , \tag{7a}$$

$$\bar{\sigma}_z^{(0)}(x,y,-h) = \frac{p(x,y)}{2} = \frac{a}{2} - \frac{b}{2}\ , \tag{7b}$$

$$\bar{\sigma}_z^{(e)}(x,y,\pm h) = -\frac{q(x,y)}{2} = -\frac{a}{2} - \frac{b}{2}\ . \tag{8}$$

The plate problem with applied boundary-forces $\bar{\sigma}_n^{(0)}(s,z)$, $\bar{\tau}_{ns}^{(0)}(s,z)$, $\bar{\tau}_{nz}^{(e)}(s,z)$ with face-forces (7a,b) will be called Problem IIa after [1], whereas the plate problem with applied boundary-forces $\bar{\sigma}_n^{(e)}(s,z)$, $\bar{\tau}_{ns}^{(e)}(s,z)$, $\bar{\tau}_{nz}^{(e)}(s,z)$ and plate forces (8) will be called Problem IIb. The Problem IIa is a pure bending problem, while the Problem IIb is a generalized plane stress problem. The plate problem we consider may be uniquely decomposed into the pair of problems (IIa,IIb) as described above.

As in [1] the Problems IIa, and IIb also split; indeed we may introduce Problems III, IV, V, and VI such that

solution III + solution IV = solution IIa,

and

solution V + solution VI = solution IIb.

This is done by seeking solutions in terms of even and odd parts, namely $u=u^e+u^0$, $\sigma_x=\sigma_x^e+\sigma_x^0$, etc. Upon substituting these into the system (5a-5c) and (1a-1f) the resulting equations separate into two independent systems, namely we obtain

Problem III

$$\text{unknowns} \begin{cases} \sigma_x^0,\ \sigma_y^0,\ \sigma_z^0,\ \tau_{xy}^0,\ u^0,\ v^0 & \text{odd in } z \\ \tau_{xy}^e,\ \tau_{yz}^e,\ w^e & \text{even in } z \end{cases}$$

$$\text{boundary conditions} \begin{cases} \bar{\sigma}_n^{(0)}(s,z),\ \bar{\tau}_{ns}^{(0)}(s,z),\ \bar{\tau}_{nz}^{(e)}(s,z) \\ \bar{\sigma}_z^{(0)}(x,y,h) = -\tfrac{1}{2}p,\ \bar{\sigma}_z^{(0)}(x,y,-h) = \tfrac{1}{2}p. \end{cases}$$

Problem IV[≠]

$$\text{unknowns} \begin{cases} \sigma_x^e,\ \sigma_y^e,\ \sigma_z^e,\ \tau_{xy}^e,\ u^e,\ v^e & \text{even in } z \\ \tau_{xy}^0,\ \tau_{yz}^0,\ w^0 & \text{odd in } z \end{cases}$$

$$\text{boundary conditions} \begin{cases} \bar{\sigma}_n^e=0,\ \bar{\tau}_{ns}^e=0,\ \bar{\tau}_{nz}^0=0 \\ \sigma_z^e(x,y,\pm h) = 0. \end{cases}$$

Problem V

$$\text{unknowns} \begin{cases} \sigma_x^e,\ \sigma_y^e,\ \sigma_z^e,\ \tau_{xy}^e,\ u^e,\ v^e & \text{even in } z \\ \tau_{xy}^0,\ \tau_{yz}^0,\ w^0 & \text{odd in } z \end{cases}$$

$$\text{boundary conditions} \begin{cases} \bar{\sigma}_n^e(s,z),\ \bar{\tau}_{ns}^e(s,z),\ \bar{\tau}_{nz}^0(s,z) \\ \bar{\sigma}_z^e(x,y,\pm h) = -\frac{\alpha}{2} \end{cases}$$

[≠]Problem IV represents a rigid-body motion.

Problem VI

unknowns $\begin{cases} \sigma_x^0,\ \sigma_y^0,\ \sigma_z^0,\ \tau_{xy}^0,\ u^0,\ v^0 & \text{odd in } z \\ \tau_{xy}^e,\ \tau_{yz}^e,\ w^e & \text{even in } z \end{cases}$

boundary conditions $\begin{cases} \bar{\sigma}_n^0=0,\ \bar{\tau}_{ns}^0=0,\ \bar{\tau}_{nz}^e=0, \\ \bar{\sigma}_z^0(x,y,\pm h) = 0. \end{cases}$

Analysis of Problem III

Since we wish to obtain asymptotic results as $h \to 0$ we introduce a new thickness variable $\zeta = z/h$, $\zeta \in [-1,1]$. Likewise, in order that stresses and displacements will not vanish or become unbounded we set as in [1]

$$\left.\begin{aligned} \bar{\sigma}_n^0 &= \bar{\sigma}_n^{(1)}(s,\zeta)h \\ \bar{\tau}_{ns}^0 &= \bar{\tau}_{ns}^{(1)}(s,\zeta)h \end{aligned}\right\} \quad \text{odd in } \zeta, \tag{9a}$$

$$\bar{\tau}_{nz}^e = \bar{\tau}_{nz}^{(2)}(s,\zeta)h^2 \qquad \text{even in } \zeta, \quad \bar{\tau}_{nz}^{(2)}(s,\pm 1)=0. \tag{9b}$$

On the top and bottom of the plate we require as Friedrichs and Dressler [1] that

$$\bar{\tau}_{xz}(x,y,\pm 1,h) = \bar{\tau}_{yz}(x,y,\pm 1,h) = 0, \tag{10a}$$

$$\bar{\sigma}_z = -\tfrac{1}{2}p^{(3)}(x,y)h^3, \tag{10b}$$

$$\bar{\sigma}_z = \tfrac{1}{2}p^{(3)}(x,y)h^3. \tag{10c}$$

In terms of the x,y,ζ coordinates our system (5a-5c), (1a-1f), (6a-6f) become

$$h\sigma_{x,x}+h\tau_{xy,y}+\tau_{xz,\zeta} = 0, \tag{11a}$$

$$h\tau_{xy,x}+h\sigma_{y,y}+\tau_{yz,\zeta} = 0, \tag{11b}$$

$$h\tau_{xz,x}+h\tau_{yz,y}+\sigma_{z,\zeta} = 0, \tag{11c}$$

$$u,_{x} = a_{11}\sigma_{x}+a_{12}\sigma_{y}+a_{13}\sigma_{z}, \tag{12a}$$

$$v,_{y} = a_{12}\sigma_{x}+a_{22}\sigma_{y}+a_{23}\sigma_{z}, \tag{12b}$$

$$w,_{\zeta} = ha_{13}\sigma_{x}+ha_{23}\sigma_{y}+ha_{33}\sigma_{z}, \tag{12c}$$

$$ha_{44}\tau_{yz} = v,_{\zeta}+hw,_{y}, \tag{12d}$$

$$ha_{55}\tau_{xy} = hw,_{x}+u,_{\zeta}, \tag{12e}$$

$$a_{66}\tau_{xy} = u,_{y}+v,_{x}, \tag{12f}$$

$$\begin{aligned}-a_{44}h^{2}\sigma_{x,xx}&+2a_{13}h^{2}\sigma_{x,yy}+2a_{12}\sigma_{x,\zeta\zeta}+(2a_{23}+a_{44})h^{2}\sigma_{y,yy}\\&+2a_{22}\sigma_{y,\zeta\zeta}+2a_{33}h^{2}\sigma_{z,yy}+(2a_{23}+a_{44})\sigma_{z,\zeta\zeta} = 0,\end{aligned} \tag{13a}$$

$$\begin{aligned}(2a_{13}+a_{55})h^{2}\sigma_{x,xx}&+2a_{11}\sigma_{x,\zeta\zeta}+2a_{23}h^{2}\sigma_{y,xx}-a_{55}h^{2}\sigma_{y,yy}\\&+2a_{12}\sigma_{y,\zeta\zeta}+2a_{33}h^{2}\sigma_{z,xx}+(2a_{13}+a_{55})\sigma_{z,\zeta\zeta} = 0,\end{aligned} \tag{13b}$$

$$\begin{aligned}(2a_{12}+a_{66})h^{2}\sigma_{x,xx}&+2a_{11}h^{2}\sigma_{x,yy}+2a_{22}h^{2}\sigma_{y,xx}\\&+(2a_{12}+a_{66})h^{2}\sigma_{y,yy}+2a_{23}h^{2}\sigma_{z,xx}+2a_{13}h^{2}\sigma_{z,yy}\\&-a_{66}\sigma_{z,\zeta\zeta} = 0,\end{aligned} \tag{13c}$$

$$\begin{aligned}[\,2a_{11}h\sigma_{x}&+(2a_{12}+a_{66})h\sigma_{y}+(2a_{13}+a_{55})h\sigma_{z}],_{y\zeta}+a_{44}h^{2}\tau_{yz,xx}\\&+a_{55}h^{2}\tau_{yz,yy}+a_{66}\tau_{vz,\zeta\zeta} = 0,\end{aligned} \tag{13d}$$

$$[(2a_{12}+a_{66})h\sigma_x+2a_{22}h\sigma_y+(2a_{23}+a_{44})h\sigma_z]_{,x\zeta}+a_{44}h^2\tau_{xz,xx}$$
$$+a_{55}h^2\tau_{xz,yy}+a_{66}\tau_{xz,\zeta\zeta} = 0, \tag{13e}$$

$$[(2a_{13}+a_{55})h^2\sigma_x+(2a_{23}+a_{44})h^2\sigma_y+2a_{33}h^2\sigma_z]_{,xy}+a_{44}h^2\tau_{xy,xx}$$
$$+a_{55}h^2\tau_{xy,yy}+a_{66}\tau_{xy,\zeta\zeta} = 0. \tag{13f}$$

Using the equations (11,12,13) we seek an interior expansion for the three displacements and six stresses. We assume each of these unknowns have asymptotic expansions of the form

$$u(x,y,\zeta,h) := \sum_{k=0}^{\infty} u^{(k)}(x,y,\zeta)h^k, \text{ etc.} \tag{14}$$

As in [ ] we obtain for the h^0 step quite directly that $\tau_{xz}^{(0)} \equiv \tau_{yz}^{(0)} \equiv \sigma_z^{(0)} \equiv v^{(0)} \equiv u^{(0)} \equiv \tau_{xy}^{(0)} \equiv 0$, and that $w^{(0)} := W(x,y)$ is independent of ζ. The equations for $\sigma_x^{(0)}$ and $\sigma_y^{(0)}$ become

$$a_{11}\sigma_x^{(0)} + a_{12}\sigma_y^{(0)} = 0,$$

$$a_{12}\sigma_x^{(0)} + a_{22}\sigma_y^{(0)} = 0,$$

which lead to $\sigma_x^{(0)} \equiv \sigma_y^{(0)} \equiv 0$ providing that

$$a_{11}a_{22}-a_{12}a_{21} = \frac{1}{E_1E_2}(1-\nu_{12}\nu_{21}) \neq 0.$$

The analysis for the h^1-step is similar to that in [ ] so we list the results

$$\tau_{xz}^{(1)} \equiv \tau_{yz}^{(1)} \equiv \sigma_z^{(1)} \equiv 0$$

$$w^{(1)} := W^{(1)}(x,y),\ v^{(1)} := V^{(1)}(x,y)\zeta,\ u^{(1)} := U^{(1)}(x,y)\zeta,$$

$$\tau_{xy}^{(1)} := T_{xy}^{(1)}(x,y) .$$

Furthermore, providing that condition (15) holds we have that

$$\sigma_x^{(1)} := \overset{(1)}{\textstyle\sum}_x(x,y)\zeta, \quad \sigma_y^{(1)} := \overset{(1)}{\textstyle\sum}_y(x,y)\zeta.$$

The h^2-step leads as usual to

$$\sigma_z^{(2)} \equiv 0, \quad \tau_{xy}^{(2)} := T_{xy}^{(2)}(x,y)\zeta, \quad \tau_{yz}^{(2)} := T_{yz}^{(2)}(x,y)(1-\zeta^2)$$

$$\tau_{xz}^{(2)} = T_{xz}^{(2)}(x,y)(1-\zeta^2), \quad v^{(2)} := V^{(2)}(x,y)\zeta,$$

$$u^{(2)} := U^{(2)}(x,y)\zeta,$$

and providing conditions (15) is met, we have also

$$\sigma_x^{(2)} = \overset{(2)}{\textstyle\sum}_x(x,y)\zeta, \quad \sigma_y^{(2)} = \overset{(2)}{\textstyle\sum}_y(x,y)\zeta,$$

plus a new result, namely

$$w^{(2)} = \tfrac{1}{2}\left(a_{13}\overset{(1)}{\textstyle\sum}_x(x,y) + a_{23}\overset{(1)}{\textstyle\sum}_y(x,y)\right)\zeta^2.$$

In terms of the function $W^{(0)}$ it is possible to represent $\overset{(1)}{\sum}_x$ and $\overset{(2)}{\sum}_y$ as

$$\overset{(1)}{\textstyle\sum}_y(x,y) = \frac{1}{a_{11}a_{22}-a_{12}^2}\left(-a_{22}W_{,xx}^{(0)} + a_{12}W_{,yy}^{(0)}\right), \tag{15a}$$

and

$$\overset{(1)}{\textstyle\sum}_y(x,y) = \frac{1}{a_{11}a_{22}-a_{12}^2}\left(-a_{11}W_{,yy}^{(0)} + a_{12}W_{,xx}^{(0)}\right) \tag{15b}$$

respectively. Expressions for $T_{xy}^{(1)}$, $T_{xz}^{(2)}$, and $T_{yz}^{(2)}$ may also be found in terms of $W^{(0)}(x,y)$; these are

$$T^{(1)}_{xy} = -\frac{2}{a_{66}} W^{(0)}_{,xy} , \tag{15c}$$

$$T^{(2)}_{xz}(x,y) = -\frac{1}{2a_{66}(a_{11}a_{22}-a_{12}^2)}\Big(a_{22}a_{66}W^{(0)}_{,xxx} + (2a_{11}a_{22}-2a_{12}^2-a_{12}a_{66})W^{(0)}_{,yyx}\Big) \tag{15d}$$

$$T^{(2)}_{yz}(x,y) = -\frac{1}{2a_{66}(a_{11}a_{22}-a_{12}^2)}\Big(a_{11}a_{66}W^{(0)}_{,yyy} + (2a_{11}a_{22}-2a_{12}^2-a_{12}a_{66})W^{(0)}_{,xxy}\Big) . \tag{15e}$$

Using these in (11c) leads to the following differential equation for $W^{(0)}$,

$$a_{22}a_{66}W^{(0)}_{,xxxx}+2(2a_{11}a_{22}-2a_{12}^2-a_{12}a_{66})W^{(0)}_{,xxyy} + a_{11}a_{66}W^{(0)}_{,yyyy} = -\frac{3}{2}\, a_{66}(a_{11}a_{22}-a_{12}^2)\, p^{(3)}(x,y) . \tag{15f}$$

We observe that equation (15f) is the equation of a thin, orthotropic plate as presented in the book of Lekhnitskii [3]. It is interesting to note that this reduces to the result of Friedrichs and Dressler for the isotropic plate.

The Boundary Layer Problem

We first expand the edge stresses in powers of the arc length variable s about s=0, and then replace s by th

$$\bar{\sigma}^{(1)}_{n}(s,\zeta)h = \bar{\sigma}^{(1)}_{n}(0,\zeta)h + \bar{\sigma}^{(1)}_{n,s}(0,\zeta)th^2 + \ldots \tag{16a}$$

$$\bar{\tau}^{(1)}_{ns}(s,\zeta)h = \bar{\tau}^{(1)}_{ns}(0,\zeta)h + \bar{\tau}^{(1)}_{ns,s}(0,\zeta)th^2 + \ldots \tag{16b}$$

$$\bar{\tau}^{(2)}_{nz}(s,\zeta)h^2 = \bar{\tau}^{(2)}_{nz}(0,\zeta)h^2 + \bar{\tau}^{(2)}_{nz,s}(0,\zeta)th^3 + \ldots \tag{16c}$$

and the face stresses as

$$\bar{\sigma}_z = \bar{\sigma}_z(x,y,\pm 1,h) = \mp p^{(3)}_{(0,0)} h^3 + \dots , \tag{17a}$$

$$\tau_{xz} = \tau_{yz} = 0 \quad \text{for} \quad \zeta = \pm 1. \tag{17b}$$

If the boundary curve B is parametrized by $x=x(s)$, $y=y(s)$ then under the transformation $\xi=x/h$, $\eta=y/h$ we obtain a new boundary curve B^* with arc length $t=s/h$, where the parametrization is now given by $\xi=\xi^*(t,h)$, $\eta=\eta^*(t,h)$. The unit tangent and normal vectors are given by $[X_s(t,h),Y_s]$, $[X_n,Y_n]$ respectively. In this notation the edge conditions may be rewritten (see Lekhnitskii, pg. 2), as

$$\bar{\sigma}^{(1)}_n(0,\zeta)h + \bar{\sigma}^{(1)}_{n,s}(0,\zeta)th^2 + \dots$$

$$= X_n^2 \overset{+}{\sigma}_x(\xi^*,\eta^*,\zeta,h) + 2X_nY_n\overset{+}{\tau}_{xy} + Y_n^2\overset{+}{\sigma}_y, \tag{18a}$$

$$\bar{\tau}^{(1)}_{ns}(0,\zeta)h + \bar{\tau}^{(1)}_{ns,s}(0,\zeta)h^2 t + \dots$$

$$= X_nY_s\overset{+}{\sigma}_z + [X_nY_s + X_sY_n]\overset{+}{\tau}_{xy} + Y_nY_s\overset{+}{\sigma}_y, \tag{18b}$$

$$\bar{\tau}^{(2)}_{nz}(0,\zeta)h^2 + \bar{\tau}^{(2)}_{nz,s}(0,\zeta)th^2 + \dots$$

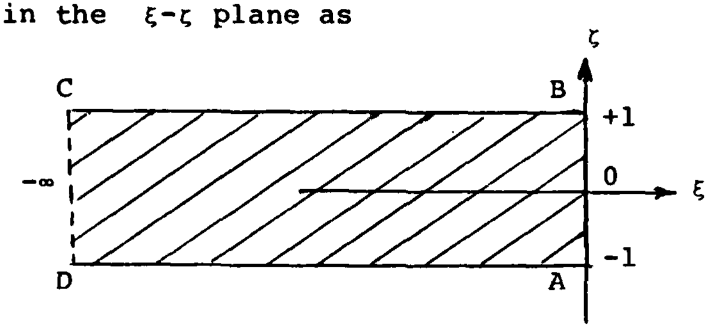
$$= X_n\overset{+}{\tau}_{xz} + Y_n\overset{+}{\tau}_{yz}. \tag{18c}$$

Here we have used the notation of [ ] to indicate that

$$\overset{+}{\sigma}(\xi,\eta,\zeta,h) := \sigma(h\xi,h\eta,\zeta,h), \text{ etc.}$$

In what follows we match interior and exterior expansions to obtain the asymptotic behavior of the various terms in the plus superscript function expansions as $\xi\to-\infty$; for details of the procedure the reader should consult [1].

The limit domain in the ξ,η,ζ space is given by $D^*:=\{(\xi,\eta,\zeta): \xi\le 0,\ |\eta|<\infty,\ -1\le\zeta\le 1\}$. We illustrate this by showing its cross-section in the $\xi-\zeta$ plane as



The segment AB corresponds to the boundary edge, whereas BC and AD correspond to the top and bottom faces respectively.

The h^0 step

In the limit domain in the ξ,η,ζ-space we obtain the following boundary conditions at the two faces and the edge

$$\overset{+}{\sigma}{}_z^{(0)} = \overset{+}{\tau}{}_{yz}^{(0)} = \overset{+}{\tau}{}_{xz}^{(0)} = 0 \quad \text{on} \quad \zeta = \pm 1 \tag{19a}$$

$$\overset{+}{\sigma}{}_x^{(0)} = \overset{+}{\tau}{}_{xy}^{(0)} = \overset{+}{\tau}{}_{xz}^{(0)} = 0 \quad \text{on} \quad \xi = 0 \tag{19b}$$

At $\xi=-\infty$ $\overset{+}{\sigma}{}^{(0)}(\xi,\eta,\zeta)=\sigma^{(0)}(0,0,\zeta)$. Furthermore, since for the interior problem $\sigma_x^{(0)}\equiv\tau_{xy}^{(0)}\equiv\tau_{xz}^{(0)}\equiv 0$ we obtain the boundary conditions

$$\overset{+}{\sigma}{}_x^{(0)} = \overset{+}{\tau}{}_{xy}^{(0)} = \overset{+}{\tau}{}_{xz}^{(0)} = 0 \quad \text{at} \quad \xi = -\infty \ . \tag{20}$$

As in [ ] one has the conditions

$$\frac{\partial \overset{+}{\sigma}{}^{(0)}}{\partial \eta} \equiv 0, \quad \frac{\partial^{2} \overset{+}{\sigma}{}^{(1)}}{\partial \eta^{2}} \equiv 0, \quad \ldots, \quad \frac{\partial^{n} \overset{+}{\sigma}{}^{(n-1)}}{\partial \eta^{n}} \equiv 0, \quad \ldots \tag{21}$$

which leads to a splitting of the stress system into a plane strain problem involving six equations in the four unknowns $\overset{+}{\sigma}{}^{(0)}_x$, $\overset{+}{\sigma}{}^{(0)}_y$, $\overset{+}{\sigma}{}^{(0)}_z$, $\overset{+}{\tau}{}^{(0)}_{xz}$ and into a torsion problem having three equations in the two unknowns $\overset{+}{\tau}{}^{(0)}_{xy}$, $\overset{+}{\tau}{}^{(0)}_{yz}$. Obtaining the asymptotic conditions for $\overset{+}{\sigma}{}^{(0)}_x$, $\overset{+}{\sigma}{}^{(0)}_y$, $\overset{+}{\sigma}{}^{(0)}_z$ from the interior solutions and integrating the zeroth under system yields

$$\overset{+}{\sigma}{}^{(0)}_y = \frac{-1}{a_{22}}\left(a_{12}\overset{+}{\sigma}{}^{(0)}_x + a_{23}\overset{+}{\sigma}{}^{(0)}_z\right).$$

Since all applied stresses are zero on the boundary of the limit domain, we conclude as in [1] that $\overset{+}{\sigma}{}^{(0)}_x$, $\overset{+}{\sigma}{}^{(0)}_z$, $\overset{+}{\tau}{}^{(0)}_{xz}$, are identically zero. This implies that $\overset{+}{\sigma}{}^{(0)}_y \equiv 0$ also. The torsion problem has vanishing boundary conditions and therefore it follows that $\overset{+}{\tau}{}^{(0)}_{xy} \equiv \overset{+}{\tau}{}^{(0)}_{yz} \equiv 0$.

The h^1 Step

We consider next the plane strain problem for $\overset{+}{\sigma}{}^{(1)}_x$, $\overset{+}{\tau}{}^{(1)}_{xz}$, $\overset{+}{\sigma}{}^{(1)}_z$, $\overset{+}{\sigma}{}^{(1)}_y$. Since all η derivatives vanish we are led to the following six equations (Here we have suppressed the +(1) super scripts in our notation.):

$$\sigma_{x,\xi} + \tau_{xz,\zeta} = 0, \tag{22a}$$

$$\tau_{xz,\xi} + \sigma_{z,\zeta} = 0, \tag{22b}$$

$$-a_{44}\sigma_{x,\xi\xi} + 2a_{12}\sigma_{x,\zeta\zeta} + 2a_{22}\sigma_{y,\zeta\zeta} + (2a_{23} + a_{44})\sigma_{z,\zeta\zeta} = 0, \tag{22c}$$

$$(2a_{13}+a_{55})\sigma_{x,\xi\xi}+2a_{11}\sigma_{x,\zeta\zeta}+2a_{23}\sigma_{y,\xi\xi}+2a_{12}\sigma_{y,\zeta\zeta}+2a_{33}\sigma_{z,\xi\xi}$$
$$+(2a_{13}+a_{55})\sigma_{z,\zeta\zeta} = 0, \tag{22d}$$

$$(2a_{12}+a_{66})\sigma_{x,\xi\xi}+2a_{22}\sigma_{y,\xi\xi}+2a_{23}\sigma_{z,\xi\xi}-a_{66}\sigma_{z,\zeta\zeta}=0, \tag{22e}$$

$$[(2a_{12}+a_{66})\sigma_x+2a_{22}\sigma_y+(2a_{23}+a_{44})\sigma_z]_{,\xi\zeta}+a_{44}\tau_{xz,\xi\xi} \tag{22f}$$
$$+a_{66}\tau_{xz,\zeta\zeta} = 0.$$

Equations (22a,b) imply the existence of a stress function $\Theta:=\Theta(\xi,\zeta)$ such that $\sigma_x=:\Theta_{,\zeta\zeta}$, $\sigma_z=:\Theta_{,\xi\xi}$, $\tau_{xz}=:-\Theta_{,\xi\zeta}$. From this equations (22c-22f) may be rewritten as

$$-a_{44}\Theta_{,\zeta\zeta\xi\xi}+2a_{12}\Theta_{,\zeta\zeta\zeta\zeta}+2a_{22}\sigma_{y,\zeta\zeta}+(2a_{23}+a_{44})\Theta_{,\xi\xi\zeta\zeta}=0, \tag{23c}$$

$$(2a_{13}+a_{55})\Theta_{,\zeta\zeta\xi\xi}+2a_{11}\Theta_{,\zeta\zeta\zeta\zeta}+2a_{23}\sigma_{y,\xi\xi}+2a_{12}\sigma_{y,\zeta\zeta}$$
$$+2a_{33}\Theta_{,\xi\xi\xi\xi}+(2a_{13}+a_{55})\Theta_{,\xi\xi\zeta\zeta} = 0, \tag{23d}$$

$$(2a_{12}+a_{66})\Theta_{,\zeta\zeta\xi\xi}+2a_{22}\sigma_{y,\xi\xi}+2a_{23}\Theta_{,\xi\xi\xi\xi}-a_{66}\Theta_{,\xi\xi\zeta\zeta}=0, \tag{23e}$$

$$[(2a_{12}+a_{66})\Theta_{,\zeta\zeta}+2a_{22}\sigma_y+(2a_{23}+a_{44})\Theta_{,\xi\xi}]_{,\xi\zeta}$$
$$-\Theta_{,\xi\zeta\xi\xi}a_{44}-a_{66}\Theta_{,\xi\zeta\zeta\zeta} = 0, \tag{23f}$$

Integration of (23f) leads to

$$a_{22}\sigma_y = -a_{12}\Theta_{,\zeta\zeta}-a_{23}\Theta_{,\xi\xi}+\phi_1(\xi)+\phi_2(\zeta)$$
$$= -a_{12}\sigma_x-a_{23}\sigma_z+\phi_1(\xi)+\phi_2(\zeta).$$

Using the interior expansions and matching, one obtains the asymptotic behavior as $\xi\to-\infty$, from which follows

$$a_{22}\sigma_y = -a_{12}\sigma_x - a_{23}\sigma_z + c_1\xi + c_2\zeta + c_3; \tag{24}$$

however, as σ_y is odd in the ζ-variable we are forced to have $c_1 = c_3 = 0$. After some further manipulation with the equations (23c-e) it is found that θ satisfies

$$(a_{22}a_{33}-a_{23}^2)\theta,_{\xi\xi\xi\xi}+(a_{22}[2a_{13}+a_{55}]-2a_{12}a_{23})\theta,_{\xi\xi\zeta\zeta}$$
$$+(a_{11}a_{22}-a_{12}^2)\theta,_{\zeta\zeta\zeta\zeta} = 0. \tag{25}$$

For the case of an isotropic material equation (25) reduces to the biharmonic equation as was shown in the paper of Friedricks and Dressler. For a material with one plane of isotropy the equation becomes

$$(1-\frac{E}{E'}\nu'^2)\theta,_{\xi\xi\xi\xi} + 2(1-\nu\nu')\theta,_{\xi\xi\zeta\zeta} + \frac{E'}{E}(1-\nu^2)\theta,_{\zeta\zeta\zeta\zeta}=0; \tag{26}$$

whereas for an orthotropic material (25) may be expressed in terms of the elastic coefficients as

$$\frac{1}{E_3}(1-\nu_{23}\nu_{32})\theta,_{\xi\xi\xi\xi}+(-\frac{2\nu_{13}}{E_1} + \frac{1}{G_{13}} - \frac{2}{E_1}\nu_{12}\nu_{23})\theta,_{\xi\xi\zeta\zeta}$$
$$+ \frac{1}{E_1}(1-\nu_{12}\nu_{21})\theta,_{\zeta\zeta\zeta\zeta} = 0. \tag{27}$$

Equations (26,27) show the three-dimensional nature of the equation which θ satisfies in that it contains elastic constants associated with the z-direction.

Recalling that $\sigma_x=:\theta,_{\zeta\zeta}$, $\sigma_z=:\theta,_{\xi\xi}$, $-\tau_{xz}=:\theta,_{\xi\zeta}$ we obtain the following boundary conditions for θ, namely

$$\left.\begin{array}{l}\Theta,_{\zeta\zeta} = \bar{\sigma}_n^{(1)}(0,\zeta) \\ \Theta,_{\xi\zeta} = 0\end{array}\right\} \quad \text{on AB} \tag{28a}$$

and

$$\Theta,_{\xi\xi} = \Theta,_{\xi\zeta} = 0 \quad \text{on BC and DA} \tag{28b}$$

On CD we obtain

$$\Theta,_{\zeta\zeta} = \overset{(1)}{\sum_x}(0,0)\zeta, \quad \text{and} \quad \Theta,_{\xi\zeta} = \Theta,_{\xi\xi} = 0. \tag{29}$$

From (24), (29) and the fact that $\sigma_y = \overset{(1)}{\sum_x}(0,0)\zeta$ on CD the expression for σ_y becomes

$$a_{22}\sigma_y = -a_{12}\Theta,_{\zeta\zeta} - a_{23}\Theta,_{\xi\zeta} + (a_{22}\overset{(1)}{\sum_x}(0,0) + a_{12}\overset{(1)}{\sum_x}(0,0))\zeta .$$

This completes the h^1-step for $\sigma_x, \sigma_z, \tau_{xz}$, and σ_y, modulo the fact that our plane-strain problem is defined only in terms of the "interior" stress problem evaluated at the edge of the plate. For further details on how to determine the proper boundary conditions for the interior problem in order to determine $\overset{(1)}{\sum_x}(0,0)$, $\overset{(1)}{\sum_x}(0,0)$ the reader is referred to [1], pp. 21-22.

We turn now to the torsion problem for $\overset{+(1)}{\tau}_{xy}$, $\overset{+(1)}{\tau}_{yz}$, as above we shall omit the +(1) superscripts.

Equations (11b), (13d), (13f) reduce to

$$\tau_{xy,\xi} + \tau_{yz,\zeta} = 0, \tag{30a}$$

$$a_{44}\tau_{yz,\xi\xi} + a_{66}\tau_{yz,\zeta\zeta} = 0, \tag{30b}$$

$$a_{44}\tau_{xy',\xi\xi} + a_{66}\tau_{xy,\zeta\zeta} = 0, \tag{30c}$$

Introducing

$$\tau_{xy} =: \phi,_{\zeta} \ , \ -\tau_{yz} =: \phi,_{\xi} \tag{31}$$

allows (30b,c) to be rewritten in the form

$$a_{44}\phi,_{\xi\xi\xi} + a_{66}\phi,_{\xi\zeta\zeta} = \zeta, \tag{32a}$$

$$a_{44}\phi,_{\zeta\xi\xi} + a_{66}\phi,_{\zeta\zeta\zeta} = 0, \tag{32b}$$

which imply that

$$a_{44}\phi,_{\xi\xi} + a_{66}\phi,_{\zeta\zeta} = 0, \tag{33a}$$

where δ is an arbitrary constant. This reduces to the Friedricks-Dressler form, $\Delta\phi=\delta$, when the material is isotropic. When there is a plane of isotropy this becomes

$$E(1+\nu')\phi,_{\xi\xi} + E'(1+\nu)\phi,_{\zeta\zeta} = \delta \ , \tag{33b}$$

and for an orthogropic material equation (33a) in terms of its elastic coefficients is

$$\frac{1}{G_{23}}\phi,_{\xi\xi} + \frac{1}{G_{12}}\phi,_{\zeta\zeta} = \delta.$$

The boundary conditions for (33a) are

$$\phi,_{\zeta} := \tau_{xy}^{(1)}(0,\eta,\zeta) = \bar{\tau}_{ns}^{(1)}(0,\zeta) \quad \text{on} \quad AB \tag{34}$$

and

$$\phi,_{\xi} := -\overset{+}{\tau}{}_{yz}^{(1)}(\xi,\eta,\pm 1,h) = 0 \quad \text{on} \quad BC \text{ and } AD. \tag{35}$$

Since $\overset{+}{\tau}{}_{xy}^{(1)} \to T_{xy}^{(1)}(0,0)\zeta$ as $\xi\to-\infty$, we have

$$\phi,_{\zeta} = T_{xy}^{(1)}(0,0)\delta \quad \text{on} \quad CD. \tag{36}$$

This boundary value problem may be solved by Fourier series as suggested in [1]. The arbitrary coefficient is determined to be $T_{xy}^{(1)}(0,0)$.

The h^2-Step

Here the boundary conditions are, as usual, the same as in [ ]; the reader is referred to equations (51-55) of this work for details. We treat next the case of plane-strain for the so-called "excess" stresses, [1], pg. 25, $\overset{+(2)}{\underline{\sigma}}_{x,\eta}$, $\overset{+(2)}{\underline{\tau}}_{xz,\eta}$, $\overset{+(2)}{\underline{\tau}}_{x,\eta}$, and $\overset{+(2)}{\underline{\sigma}}_{z,\eta}$. As has been our custom we introduce a stress function $\Gamma(\xi,\zeta)$ such that

$$\Gamma_{,\zeta\zeta} := \underline{\sigma}_{x,\eta},\quad \Gamma_{,\xi\xi} := \underline{\sigma}_{z,\eta},\quad \Gamma_{,\xi\zeta} := -\underline{\tau}_{xz,y}. \tag{37}$$

Then Γ is seen to satisfy

$$\begin{aligned}(a_{22}a_{33}-a_{23}^2)\Gamma_{,\xi\xi\xi\xi} &+ (a_{22}[2a_{13}+a_{55}]-2a_{12}a_{23})\Gamma_{,\xi\xi\zeta\zeta}\\ &+(a_{11}a_{22}-a_{12}^2)\Gamma_{,\zeta\zeta\zeta\zeta} = 0.\end{aligned} \tag{38}$$

Furthermore, $\underline{\sigma}_{y,\eta}$ is found to be

$$\begin{aligned}a_{22}\underline{\sigma}_{y,\eta} &= -a_{12}\underline{\sigma}_{x,\eta}-a_{23}\underline{\sigma}_{z,\eta}+\phi_1(\xi)+\phi_2(\zeta)\\ &= -a_{12}\Gamma_{,\zeta\zeta}-a_{23}\Gamma_{,\xi\xi}+\phi_1(\xi)+\phi_2(\zeta).\end{aligned} \tag{39}$$

From the fact that $\underline{\sigma}_{y,\eta}$ is odd in ζ and all the stresses vanish at infinity we conclude $\phi_1 \equiv \phi_2 \equiv 0$.

We turn next to the "excess" torsion problem for $\overset{+(2)}{\underline{\tau}}_{xy,\eta}$ and $\overset{+(2)}{\underline{\tau}}_{yz,\eta}$ and following our previous notational abreviation drop the +(2) superscript. We have from (30a-c)

$$\tau_{xy,\eta\xi} + \tau_{yz,\eta\zeta} = 0, \tag{40a}$$

$$a_{44}\tau_{yz,\eta\xi\xi} + a_{66}\tau_{yz,\eta\zeta\zeta} = 0, \tag{40b}$$

$$a_{44}\tau_{xy,\eta\xi\xi} + a_{66}\tau_{xy,\eta\zeta\zeta} = 0. \tag{40c}$$

Setting $\phi_{,\zeta} := \tau_{xy,\eta}$, $\phi_{,\xi} := -\tau_{yz,\eta}$, we obtain after integration and considering the behavior as $\xi \to -\infty$, that

$$a_{44}\phi_{,\xi\xi} + a_{66}\phi_{,\zeta\zeta} = 0. \tag{41}$$

Again the "excess" stress can be determined by solving (41) with ϕ prescribed over the boundary of our fundamental strip. We must now consider the system of "excess" stresses themselves, namely functions of the form

$$\overset{+(2)}{\underline{\sigma}_i}(\xi,\eta,\zeta) := \overset{+(2)}{\underline{\sigma}_{i,\eta}}(\xi,\zeta)\eta + A_i(\xi,\eta). \tag{42}$$

The equations for the $A_i(\xi,\eta)$ are obtained by direct substitution into the nine elasticity equations in the ξ,η,ζ defining the (+)2 excess stresses. As before, the system splits into two independent sets one which we refer to, after Friedrichs-Dressler, as "quasi-plane-strain" and the other as "quasi-torsion."

Quasi-plane-strain

From the equations (11a), (11c), (13a), (13b), (13c), and (13e) we obtain

$$A_{x,\xi} + A_{xz,\zeta} = -\overset{+(2)}{\tau}_{xy,\eta} =: -\phi_{,\zeta}, \tag{43a}$$

$$A_{xz,\xi} + A_{z,\zeta} = -\overset{+(2)}{\tau}_{yz,\eta} =: \phi_{,\xi}, \tag{43b}$$

$$-a_{44}A_{x,\xi\xi}+2a_{12}A_{x,\zeta\zeta}+2a_{22}A_{y,\zeta\zeta}+(2a_{23}+a_{44})A_{z,\zeta\zeta} = 0, \tag{43c}$$

$$\begin{gathered}(2a_{13}+a_{55})A_{x,\xi\xi}+2a_{11}A_{x,\zeta\zeta}+2a_{23}A_{y,\xi\xi}+2a_{12}A_{y,\zeta\zeta}\\ +2a_{33}A_{z,\xi\xi}+(2a_{13}+a_{55})A_{z,\zeta\zeta} = 0,\end{gathered} \tag{43d}$$

$$(2a_{12}+a_{66})A_{x,\xi\xi}+2a_{22}A_{y,\xi\xi}+2a_{23}A_{z,\xi\xi}-a_{66}A_{z,\zeta\zeta} = 0, \tag{43e}$$

$$[(2a_{12}+a_{66})A_x+2a_{22}A_y+(2a_{23}+a_{44})A_z]_{,\xi\zeta}+a_{44}A_{xz,\xi\xi} \\ +a_{66}A_{xz,\zeta\zeta} = 0. \tag{43f}$$

For an orthotropic material these simplify somewhat, namelv equations (43c-f) become

$$(1+\nu')A_{x,\xi\xi}+(1+\nu)\frac{E'}{E}A_{x,\zeta\zeta}-\frac{E'}{E}(A_{x,\zeta\zeta}+A_{y,\zeta\zeta})-A_{z,\zeta\zeta}=0, \tag{44c}$$

$$(1+\nu')A_{y,\xi\xi}+(1+\nu)\frac{E'}{E}A_{y,\zeta\zeta}-(A_x+A_y+A_z)_{,\xi\zeta}-\frac{E'}{E}(A_{x,\zeta\zeta}+A_{y,\zeta\zeta}) \\ -A_{z,\zeta\zeta} = 0, \tag{44d}$$

$$(1+\nu')A_{z,\xi\xi}+(1+\nu)\frac{E'}{E}A_{z,\zeta\zeta}-\frac{E'}{E}(A_x+A_y+\frac{E}{E'}A_z)_{,\xi\xi} = 0, \tag{44e}$$

$$[A_x+A_y+\frac{E}{E'}A_z]_{,\xi\zeta}+(1+\nu')\frac{E}{E'}A_{xz,\xi\xi}+(1+\nu)A_{xz,\zeta\zeta} = 0, \tag{44f}$$

which in turn reduce to that of [1] for the isotropic case. The boundary conditions on the quasi-plane-strain are the same as those in Friedrichs-Dressler, pp. 28-29.

Quasi-torsion

From equations (11b), (13d), (13f) we obtain

$$A_{xy,\xi} + A_{yz,\zeta} = -\sigma_{y,\eta} = \frac{a_{12}}{a_{22}}\Gamma_{,\zeta\zeta} + \frac{a_{23}}{a_{22}}\Gamma_{,\xi\xi}, \tag{45a}$$

$$a_{44}A_{yz,\xi\xi}+a_{66}A_{yz,\zeta\zeta}=-[(2a_{11}-\frac{a_{12}}{a_{22}}(2a_{12}+a_{66})\Gamma_{,\zeta\zeta\zeta} \\ +(2a_{13}+a_{55}-\frac{a_{23}}{a_{22}}(2a_{12}+a_{66}))\Gamma_{,\xi\xi\zeta}], \tag{45b}$$

$$a_{44}A_{xy,\xi\xi}+a_{66}A_{xy,\zeta\zeta} = -[(2a_{13}+a_{55}-\frac{a_{12}}{a_{22}}(2a_{23}+a_{44}))\Gamma,_{\zeta\zeta\xi}$$
$$+(2a_{33}-\frac{a_{23}}{a_{22}}(2a_{23}+a_{44}))\Gamma,_{\xi\xi\xi}. \quad (45c)$$

Equations (45b,c) reduce to the Friedrichs-Dressler case for isotropic materials, namely we have

$$\Delta A_{yz} = -\Delta\Gamma,_{\zeta}, \quad (45b')$$

and

$$\Delta A_{xy} = -\Delta\Gamma,_{\xi}. \quad (45c')$$

References

1. Friedrichs, K. O. and Dressler, R. F.,: A boundary-layer theory for elastic plates, Comm. Pure Applied Math., (1) (1961), 1-33.

2. Lackman, L. M. and Pagano, N.J.,: On the prevention of delamination in composite laminates in AIAA, ASME, SAE 15th Structures, Structural Dynamics and Materials Conference paper No 74-355.

3. Lekhnitskii, S. G.: Anisotropic Plates, 2nd Edition, Gordon and Breach, New York (1968).

4. Pagano, N. J.: On the calculation of interlaminar normal stress in composite laminate, J. Composite Materials, 8 (1974) pp. 65-82.

5. Pipes, R. B. and Pagano, N. J.: Interlaminar stresses in composite laminates under uniform axial extensions, J. Composite Materials, 4 (1970) p. 538-

6. Puppo, A. H., and Evensen, H. A.: Interlaminar shear in laminated composites under generalized plane stress, J. Composite Materials 4 (1970) pp. 204-220.

7. Tang, S.: A boundary layer theory - Part I: Laminated Composites in Plane Stress, J. Composite Materials 9 (1975) pp. 33-41, Part II: Extension of laminated finite strip, (same journal issue) pp.42-52.